RADIATIVE HEAT TRANSFER: NO TEMPERATURE JUMP AT THE MEDIA BOUNDARY

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Analyzed is the problem of radiative heat transfer in a plane layer of a gray medium. It is found that in media where kL > 1 there is no jump of molecular temperature at the boundary surfaces.

A study of problems concerning the radiative mode of heat transfer is generally fraught with serious mathematical difficulties. Significant simplifications are possible with models of an optically dense medium, based on the Rosseland approximation. The use of such a model for an analysis of the interaction between radiation and a reflecting surface leads to incorrect results, however, especially for media with a moderate or a weak absorption.

A definite improvement can be achieved by using a Rosseland model together with the hypothesis of a temperature jump at the boundary surface, this hypothesis having recently received wide recognition [1, 5]. It becomes relatively simple to determine the thermal radiation flux on this basis, but to calculate the temperature field, especially near the boundary surface, on the basis of a temperature jump is entirely impossible.

The inconsistency of the said hypothesis will be demonstrated here on the example of radiative heat transfer in a plane layer of a gray medium with a strong or a moderate absorption (kL > 1).

The method of adjoint asymptotic expansions [2], which has been recently developed in fluid mechanics, can be successfully applied to problems of radiative heat transfer in gray and in selective media.

This method yields expressions describing the intensity distribution in the boundary layer and far away from it, and by adjugation one can find a uniformly applicable expression for the radiation flux.

It will be assumed that the surfaces bounding the plane layer have only two properties: transmittivity and reflectivity. The ambient media on both sides of the plane layer have different temperatures T_1 and T_2 respectively. Those boundary surfaces receive external radiation not necessarily in equilibrium with its medium (Fig. 1).

We will assume further that 1) the boundary surfaces are gray, and have diffusive properties, 2) the hypothesis of local thermodynamic equilibrium is applicable, 3) the refractive index of the media is independent of the frequency and equal to unity, and 4) the radiation is one-dimensional.

The intensity field of the given layer is described by equations

$$\varepsilon m \frac{dI^+}{dx} = -I^+ + B(x); \quad 0 \leqslant m \leqslant 1,$$
$$\varepsilon m \frac{dI^-}{dx} = -I^- + B(x); \quad -1 \leqslant m \leqslant 0 \tag{1}$$

and the boundary conditions

$$x = 0, \qquad I^+ = I^+(0),$$
$$x = L, \qquad I^- = I^-(L). \tag{2}$$

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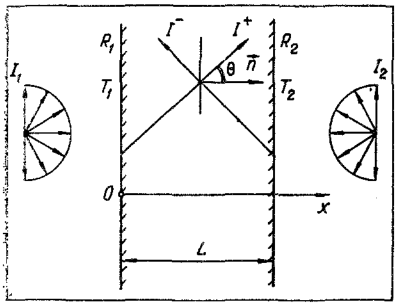

Fig. 1

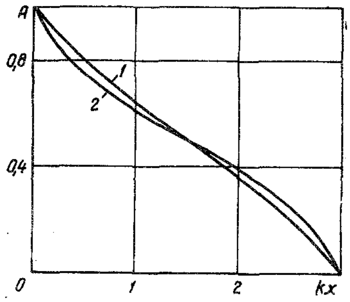

Fig. 2

Fig. 1. Schematic diagram of the problem.

Fig. 2. Emissivity profile of a plane layer of a gray medium with kL = 3 and $R_1 = R_2$, according to formula (20): 1) $R_1 = R_2 = 0$, 2) $R_1 = R_2 = 0.9$; $A = [B(x)-B(L)]/[B(0)-B(L)]$.

Considering that $\varepsilon < 1$ in optically dense media, one may seek the solution to these equations in the form of such asymptotic series (outer expansions):

$$I^+ = I_0^+ + \varepsilon I_1^+ + \varepsilon^2 I_2^+ + \dots,$$
$$I^- = I_0^- + \varepsilon I_1^- + \varepsilon^2 I_2^- + \dots \tag{3}$$

Inserting these series into (1) yields the unknown functions. As a result, we have

$$I^+ = I^- = B(x) - \varepsilon m \frac{dB}{dx} + O(\varepsilon^2). \tag{4}$$

It should be emphasized that function I^+, determined by series (4), does not satisfy its boundary conditions at x = 0. Obviously, such a series is unsuitable near this boundary (inhomogeneity region). The same applies to the properties of series (4) for function I^- near the boundary x = L.

Such a situation arises when the derivative drops out in the zeroth approximation, after series (3) are inserted into Eq. (1).

We will now search for functions I^+ and I^- uniformly applicable in the inhomogeneity region. For this purpose, we change the variables and their functions in (1) as follows

$$X_1 = \frac{x}{\varepsilon}; \quad X_2 = \frac{L-x}{\varepsilon}; \quad I^+(\varepsilon, x) = J^+(\varepsilon, X_1);$$
$$I^-(\varepsilon, x) = J^-(\varepsilon, X_2). \tag{5}$$

Equations (1) will now be rewritten as

$$m \frac{dJ^+}{dX_1} = -J^+ + B(\varepsilon X_1),$$
$$m \frac{dJ^-}{dX_2} = -J^- + B(\varepsilon X_2). \tag{6}$$

The boundary conditions remain the same.

We will seek the solutions to these equations in the form of the following asymptotic series (inner expansions):

$$J^+ = J_0^+ + \varepsilon J_1^+ + \varepsilon^2 J_2^+ + \dots,$$
$$J^- = J_0^- + \varepsilon J_1^- + \varepsilon^2 J_2^- + \dots \tag{7}$$

Considering that

$$B(\varepsilon X_1) = B(0) + B'(0) X_1 \varepsilon + \dots,$$
$$B(\varepsilon X_2) = B(L) - B'(L) X_2 \varepsilon + \dots, \tag{8}$$

with the prime sign denoting a derivative with respect to x, we can insert the inner expansions into the corresponding equations (6) and boundary conditions (2), and thus find the unknown functions there:

$$
\begin{aligned}
J^{+} &= B(0) + [I^{+}(0) - B(0)] \exp\left(-\frac{X_1}{m}\right) \\
&+ \varepsilon m B'(0)\left[\frac{X_1}{m} - 1 + \exp\left(-\frac{X_1}{m}\right)\right] + O(\varepsilon^2); \\
J^{-} &= B(L) + [I^{-}(L) - B(L)] \exp\frac{X_2}{m} \\
&- \varepsilon m B'(L)\left[\frac{X_2}{m} + 1 - \exp\frac{X_2}{m}\right] + O(\varepsilon^2).
\end{aligned}
\tag{9}
$$

Here B(0) and B(L) denote the emission power density of a black body at the respective layer boundaries.

These expressions describe accurately enough the radiation field near the respective boundary surfaces, i. e., where the outer expansions (4) are inoperative.

We will now use the method of additive superposition [2] for a composite expansion uniformly applicable over the given region:

$$
\begin{aligned}
I^{+} &= B(x) - \varepsilon m B'(x) + [I^{+}(0) - B(0)] \exp\left(-\frac{x}{\varepsilon m}\right) \\
&+ \varepsilon m B'(0) \exp\left(-\frac{x}{\varepsilon m}\right) + O(\varepsilon^2), \\
I^{-} &= B(x) - \varepsilon m B'(x) + [I^{-}(L) - B(L)] \exp\left(\frac{L-x}{\varepsilon m}\right) \\
&+ \varepsilon m B'(L) \exp\frac{L-x}{\varepsilon m} + O(\varepsilon^2).
\end{aligned}
\tag{10}
$$

Both $I^{+}(0)$ and $I^{-}(L)$ are found from the balance of radiant energy at the boundary surfaces. Defining the radiation flux as

$$
q_r = q^{+} - q^{-} = 2\pi \int_0^1 m I^{+} dm - 2\pi \int_0^{-1} m I^{-} dm,
\tag{11}
$$

one can also write the equalities

$$
\begin{aligned}
q^{+}(0) &= \pi I^{+}(0) = \pi (1 - R_1) I_1 + R_1 q^{-}(0); \\
q^{-}(L) &= \pi I^{-}(L) = \pi (1 - R_2) I_2 + R_2 q^{+}(L).
\end{aligned}
\tag{12}
$$

If it is assumed that the region affected by a boundary surface (the boundary layer) extends from the wall through a distance at which the exponential terms in expressions (10) become negligible, then, with the boundary layers not contiguous, $q^{-}(0)$ and $q^{+}(L)$ can be determined from the outer expansions (3):

$$
\begin{aligned}
q^{-}(0) &= \pi B(0) + \frac{2\pi}{3k} B'(0), \\
q^{+}(L) &= \pi B(L) - \frac{2\pi}{3k} B'(L).
\end{aligned}
\tag{13}
$$

After inserting (13) into (12), we have

$$
\begin{aligned}
I^{+}(0) &= (1 - R_1) I_1 + R_1 B(0) + \frac{2}{3k} R_1 B'(0), \\
I^{-}(L) &= (1 - R_2) I_2 + R_2 B(L) - \frac{2}{3k} R_2 B'(L).
\end{aligned}
\tag{14}
$$

Equalities (10), (11), and (14) yield an expression for a one-dimensional radiation flux through a plane layer of a medium:

TABLE 1. Dimensionless Radiation Flux

kL	R	$\frac{q_r}{\pi[B(0)-B(L)]}$ according to [3]	according to [4]	according to formula (21)
0,1	0	0,916	0,916	0,833
	0,1	0,761	0,762	0,697
	0,5	0,330	0,324	0,301
	0,9	0;052	0,0523	0,0492
1,0	0	0,553	0,553	0,533
	0,1	0,493	0,494	0,474
	0,5	0,265	0,262	0,250
	0,9	0,050	0,0505	0,0476
10,0	0	0,109	0,117	0,116
	0,1	0,107	0,113	0,113
	0,5	0,090	0,0945	0,0930
	0,9	0,038	0,0376	0,0360

$$q_r = 2\pi\left\{(1-R_1)[I_1 - B(0)] + \frac{2}{3k} R_1 B'(0)\right\} E_3(kx)$$
$$- 2\pi\left\{(1-R_2)[I_2 - B(L)] - \frac{2}{3k} R_2 B'(L)\right\} E_3[k(L-x)]$$
$$-\frac{4\pi}{3k} B'(x) + \frac{2\pi}{k} B'(0) E_4(kx) + \frac{2\pi}{k} B'(L) E_4[k(L-x)] + O(\varepsilon^2).$$

This expression differs from the well known Rosseland formula for a radiation flux through optically dense media by a few extra terms which account for the effects of boundary surfaces and of external radiation.

Steady-state heat radiation is described by the equation

$$\frac{dq_r}{dx} = 0. \tag{16}$$

We will consider only the case where the external radiation fluxes I_1 and I_2 are in equilibrium with their respective media. Then

$$I_1 = B(0) = \frac{\sigma}{\pi} T_1^4; \quad I_2 = B(L) = \frac{\sigma}{\pi} T_2^4. \tag{17}$$

Expressions (15)-(17) yield an equation which describes the radiation field in a plane layer of a gray medium:

$$\frac{4\pi}{3k} B''(x) = -\frac{4\pi}{3} R_1 B'(0) E_2(kx) + \frac{4\pi}{3} R_2 B'(L) E_2[k(L-x)]$$
$$- 2\pi B'(0) E_3(kx) + 2\pi B'(L) E_3[k(L-x)]. \tag{18}$$

Such a problem, when formulated in terms of a model of an optically dense medium (in the Rosseland approximation), reduces to Eq. (18) without the right-hand side. As a consequence, errors are incurred in the calculation of the temperature field near the surface during heat radiation.

Equation (18) must be solved for the boundary conditions

$$x = 0, \quad B(0) = \frac{\sigma}{\pi} T_1^4; \quad x = L, \quad B(L) = \frac{\sigma}{\pi} T_2^4. \tag{19}$$

The solution is written as

$$\frac{B(x) - B(L)}{B(0) - B(L)} = C\left\{\frac{3}{2}\frac{R_1}{1-R_1} E_4(kx) - \frac{3}{2}\frac{R_2}{1-R_2} E_4[k(L-x)]\right.$$
$$+ \frac{9}{4}\frac{E_5(kx)}{1-R_1} - \frac{9}{4}\frac{E_5[k(L-x)]}{1-R_2} + \frac{3}{4} k(L-x)$$
$$\left. + \frac{1}{2}\frac{R_2 + \frac{9}{8}}{1-R_2}\right\}, \tag{20}$$

where

$$C = \frac{q_r}{\pi [B(0) - B(L)]} = \frac{1}{\frac{17}{16}\frac{1}{1-R_1} + \frac{17}{16}\frac{1}{1-R_2} - 1 + \frac{3}{4}kL}. \quad (21)$$

For comparison, we have tabulated values of the dimensionless radiation flux according to formula (21) and according to data in [3] and [4]. As expected, formula (21) becomes less accurate at lower values of the product kL. The exact solution is approached closely at moderate and large values of kL.

An emissivity profile layer with kL = 3 and with various reflectivities of the boundary surfaces is shown in Fig. 2. The distribution of molecular temperature, uniquely related to the emissivity of the medium, is continuous everywhere within the layer — including its boundaries.

A Rosseland model is valid for a region far away from the boundary, separated from it by a distance equal to at least the mean-free-path length of a photon [5]. Combining expressions (4) and (11) will lead to to the same conclusion. Consequently, the well known Rosseland formula for a radiation flux becomes inoperative near a boundary surface, as does the external expansion (4).

Heat radiation near a boundary surface does not follow the laws of an optically dense medium. Expression (15) contains exponential integrals which account for boundary effects. Their contribution becomes more significant with weaker absorption in the layer.

The presence of these integrals is a consequence of the inner expansions (9), which describe the transmission of radiant energy near a boundary at distance from it equal to several mean free paths of a photon. An unjustified extension of the Rosseland model to the boundary region has led to the erroneous conclusion concerning a temperature jump during radiative heat transmission through a thermally nonconductive medium. An analysis by the method of adjoint asymptotic expansions shows, on the other hand, that no such jump of molecular temperature occurs at the boundary.

The balance of radiant energy at a boundary surface, as derived in [6], establishes only a jump of radiation temperature but not of molecular temperature.

The results presented here are valid for media where kL > 1.

NOTATION

I^+, I^-	are the radiation intensities along the x-axis in the positive and in the negative direction respectively;
$m = \cos\theta$;	
R_1, R_2	are the reflectivities of the two boundary surfaces;
L	is the layer thickness;
k	is radiative absorptivity;
$B(x) = \sigma/\pi T^4(x)$	is the emissivity of a black body;
T	is the temperature;
$E_n(x)$	is the exponential integral.

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